Problem of the Week

Problem E

Make Gauss Proud

Johann Carl Friedrich Gauss was a mathematician who lived from 1777 to 1855. He made major contributions to number theory and algebra, to name just a few. Some of the earliest stories about Gauss deal with determining the sum of sequences of numbers. Our problem today deals with a long sequence of numbers. So, make Gauss proud as you work with this problem.

A sequence consists of 2018 terms. Each term after the first term is 1 greater than the previous term. The sum of the 2018 terms is 27 243.

Determine the sum of the odd numbered terms. That is, determine the sum of every second term starting with the first term and ending with the second last term.

Johann Carl Friedrich Gauss
April 30, 1777 - February 23, 1855

Some helpful information about sequences is included on the next page. You may or may not wish to refer to it.

A sequence is made up of terms like t_1, t_2, t_3, $\cdots$, t_n. The subscript indicates the position of the term in the sequence. For example, t_8 would represent the term in the eighth position in the sequence and t_n represents the general term in the sequence.

A series is the sum of the terms of a sequence. So $t_1 + t_2 + t_3 + \cdots + t_n$ would represent the sum of the first n terms.

The following information *may* be helpful in the solution of the problem.

An *arithmetic sequence* is a sequence in which each term after the first is obtained from the previous term by adding a constant. For example, 3,5,7,9 is an arithmetic sequence with four terms and constant difference 2.

The general term of an arithmetic sequence is $t_n = a + (n-1)d$, where a is the first term, d is the constant difference and n is the number of terms.

The sum, S_n, of the first n terms of an arithmetic sequence can be found using either $S_n = \frac{n}{2}[2a + (n-1)d]$ or $S_n = n\left(\frac{t_1 + t_n}{2}\right)$, where t_1 is the first term of the sequence and t_n is the n^{th} term of the sequence.

The following example is provided to verify the accuracy of the formulas and to illustrate their use.

For the arithmetic sequence 3,5,7,9, $a = t_1 = 3$, $d = 2$, $n = 4$ and $t_n = t_4 = 9$.

$$S_n = 3 + 5 + 7 + 9 = 24$$

$$S_n = \frac{n}{2}[2a + (n-1)d] = \frac{4}{2}[2(3) + (3)2] = 2[12] = 24$$

$$S_n = n\left(\frac{t_1 + t_n}{2}\right) = 4\left(\frac{3+9}{2}\right) = 4(6) = 24$$

Gauss developed a specific formula for the sum of the first n positive integers.

$$1 + 2 + 3 + \cdots + (n-1) + n = \frac{n(n+1)}{2}$$

For example, in honour of the current year, the sum of the positive integers from 1 to 2017 is

$$1 + 2 + 3 + \cdots + 2016 + 2017 = \frac{2017(2018)}{2} = 2\,035\,153$$